JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 19, No. 4, July–August 1996

# Synthesis of Optimal $H_\infty$ Controllers via $H_2$-Based Loop-Shaping Design

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**Synthesis of optimal $H_\infty$ controllers is formulated as a loop-shaping problem where the desired closed-loop shape to be pursued is a uniform frequency response of the largest singular value. The weighted $H_2$ optimization technique used in the linear quadratic Gaussian design with loop transfer recovery is exploited in the loop-shaping procedures to generate a sequence of $H_2$ controllers converging to the optimal $H_\infty$ controller. The resulting optimal $H_\infty$ controller not only has the inherent robust property due to $H_\infty$ criterion but also possesses the nice $H_2$ control structure, being easy to compute and implement. A fighter example and a large space structure example are demonstrated to show that the numerical accuracy of the present $H_2$-based $H_\infty$ synthesis is comparable to the conventional $H_\infty$ approach, i.e., $\gamma$-iteration, but with reduced computational efforts.**

## Nomenclature

| | |
|---|---|
| $\|A(s)\|_2$ | $= [(1/2\pi)\int_{-\infty}^{\infty} \mathrm{tr}[A^*(j\omega)A(j\omega)]\,\mathrm{d}\omega]^{1/2}$ |
| | $= [(1/2\pi)\int_{-\infty}^{\infty} \sum_i \sigma_i^2[A(j\omega)]\,\mathrm{d}\omega]^{1/2}$ |
| $\|A(s)\|_\infty$ | $= \sup_{-\infty<\omega<\infty} \bar{\sigma}[A(j\omega)]$ |
| $G$ | = nominal plant |
| $K_i$ | = optimal $H_2$ controller at the $i$th iteration |
| $M$ | $= F_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$ |
| $P$ | = augmented plant, $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ |
| $R$ | = control sensitivity function, $K(I + GK)^{-1}$ |
| $S$ | = output sensitivity function, $(I + GK)^{-1}$ |
| $T$ | = complementary sensitivity function, $GK(I + GK)^{-1}$ |
| $W_i(s)$ | = scalar weighting function at the $i$th iteration |
| $W_R$, $W_S$, $W_T$ | = weighting functions associated with $R$, $S$, $T$, respectively |
| $\gamma_i(\omega)$ | $= \bar{\sigma}[W_i F_l(P, K_i)(j\omega)]$ |
| $\zeta_i(\omega)$ | $= \bar{\sigma}[F_l(P, K_i)(j\omega)]$ |
| $\lambda_i$ | $= \|W_i F_l(P, K_i)\|_2$ |
| $\bar{\sigma}(M)$ | = largest singular value of $M$ |
| $\sigma_i(M)$ | = $i$th largest singular value of $M$ |

## Introduction

IT has been recognized for long that the $H_\infty$ approach to control system is very appropriate for the optimization of stability and disturbance rejection robustness properties, whereas the linear quadratic Gaussian (LQG) type of cost function is, on the other hand, often a more practical criterion for minimizing tracking errors or control signal variations, because of reference input changes. Therefore, to quantitatively demonstrate design tradeoffs, the simultaneous treatment of both $H_2$ and $H_\infty$ performance criteria becomes indispensable. In conjunction with this consideration, mixed $H_2/H_\infty$ problems, such as LQG control with an $H_\infty$ performance bound[1,2] and mixed $H_2$ and $H_\infty$ performance objectives,[3,4] have been proposed and solved in the literature.

The purpose of this paper is not to solve the mixed $H_2/H_\infty$ control problems; instead, we will show that a LQG controller is itself an optimal $H_\infty$ controller. The main idea we want to point out here is that arbitrary optimal $H_\infty$ control problems for linear time invariant (LTI) systems can be solved by LQG controllers with proper selection of frequency-dependent weights. Consider the standard feedback framework for multivariable LTI systems in Fig. 1, where $P(s)$ is the given augmented plant and $K(s)$ is the controller to be designed. The synthesis of optimal $H_\infty$ controllers is to solve the following optimization problem:

$$\inf_{K \text{stabilizing}} \|F_l(P, K)\|_\infty = \inf_{K \text{stabilizing}} \sup_\omega \bar{\sigma}[F_l(P, K)(j\omega)] \tag{1}$$

where the lower linear fractional transformation (LFT) $F_l(P, K)$ is the transfer function matrix from the exogenous input $w$ to the penalized output $z$. Let $K_0$ be the corresponding optimal $H_\infty$ controller; then it is well known that $K_0$ possesses the following all-pass property:

$$\bar{\sigma}[F_l(P, K_0)(j\omega)] = \text{const} \qquad \forall\omega \tag{2}$$

This all-pass property is very useful in characterizing optimal $H_\infty$ solutions. There are many other optimization problems possessing this all-pass property.[5] In this paper, we introduce loop-shaping design to construct all-pass functions based on the $H_2$-optimization technique. The significance of this approach is that the resulting optimal $H_\infty$ controllers can be assigned a priori with LQG structure.

The first attempt to obtain optimal $H_\infty$ controllers by using optimization criterion other than $H_\infty$ norm was made by Kwakernaak.[6] The key point of his result is because of the following observation. Suppose that there exists a nonnegative, rational, strictly proper matrix $W(j\omega)$, such that when the cost function

$$\int_{-\infty}^{\infty} \mathrm{tr}[W(j\omega)F_l(P, K)(j\omega)]\,\mathrm{d}\omega$$

is minimized by the controller $K_0$ with $F_l(P, K_0)(j\omega)$ being all pass, then $K_0$ also minimizes $\|F_l(P, K)\|_\infty$. As shown in the next section, this result can be generalized to the form that there exists a frequency-dependent weight $W$ such that

$$\arg\inf_K \|W F_l(P, K)\|_2 = \arg\inf_K \|F_l(P, K)\|_\infty \tag{3}$$

The left-hand side of Eq. (3) is known as the frequency-weighted LQG problem,[7–9] which is intimately related to the LQG/loop

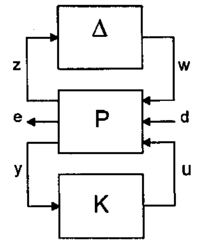


**Fig. 1 General framework of feedback control system.**

Received July 25, 1995; revision received April 2, 1996; accepted for publication April 4, 1996. 

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transfer recovery (LTR) technique.[10] Equation (3) reveals the equivalence between the $H_\infty$-optimization problem and the frequency-weighted $H_2$-optimization problem. For mixed sensitivity problems, Kwakernaak[6,11] and Grimble[12] derived the frequency-dependent weight $W(s)$ and obtained the corresponding optimal $H_\infty$ controller $K_0(s)$ in Eq. (3). Here we will go one step further by showing in the next section that for any control problems formulated by the LFT $F_l(P, K)$, there always exists a frequency-dependent weight $W(s)$ such that the equivalence in Eq. (3) is achieved.

The optimal $H_\infty$ controller $K_0$ obtained from the weighted $H_2$-optimization process not only has the inherent robust property because of the $H_\infty$ criterion but also possesses the nice $H_2$ control structure, being easy to compute and implement. The motivation of this approach is to recall the role played by weighted $H_2$ in the scenario of LQG synthesis with LTR (LQG/LTR) in which weighting functions are adjusted such that the prescribed shape of the open-loop response are met. McFarlane and Glover[13] proposed a loop-shaping design procedure using $H_\infty$ optimization, where in contrast to open-loop shaping, closed-loop objectives are specified in terms of the requirements on the singular values of the closed-loop transfer functions. In conjunction with these observations, we can interpret the problem raised by Eq. (3) in a loop-shaping manner: given the all-pass requirement (2) on the largest singular value of $F_l(P, K)(j\omega)$, find a loop-shaping design procedure using $H_2$ optimization to meet this requirement. In other words, we want to shape the $H_2$ norm of the weighted closed-loop transfer function $WF_l(P, K)$ by adjusting the weight $W$ such that the frequency response of the largest singular value of $F_l(P, K)$ becomes uniform.

## $H_2$-Based Loop-Shaping Design

Consider the feedback control system shown in Fig. 2, where $w = [d \;\; n \;\; r]$ is the exogenous input consisting of process disturbance $d$, measurement noise $n$, and reference command $r$; also, $z = [z_1 \;\; z_2 \;\; z_3]$ is the penalized output comprising weighted tracking error $z_1$, weighted control signal $z_2$, and weighted closed-loop response $z_3$; and $y$ is measurement and $u$ is control signal. The input–output relation can be written as

$$\begin{bmatrix} z \\ y \end{bmatrix} = P \begin{bmatrix} w \\ u \end{bmatrix} \tag{4}$$

where $P$ is the augmented plant given by

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{ccc|c} -W_SG & -W_S & W_S & -W_SG \\ 0 & 0 & 0 & W_R \\ W_TG & 0 & 0 & W_TG \\ \hline -G & -I & I & -G \end{array}\right]$$

The transfer function from the exogenous input $w$ to the penalized output $z$ is found by combining Eq. (4) with the control law $u = K(s)y$, leading to $z = F_l(P, K)w = [P_{11} + P_{12}K(I - P_{22}K)^{-1} P_{21}]w$. The optimal $H_\infty$ control problem is to find controller $K$ to minimize the following cost function:

$$\sup_{\|w\|_2 \le 1} \|z\|_2 = \|F_l(P, K)\|_\infty \tag{5}$$

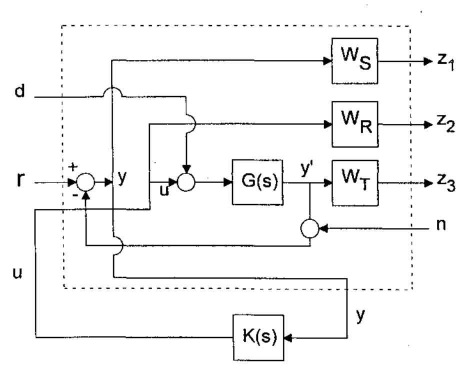


**Fig. 2 Block diagram of mixed sensitivity problem.**

Here we focus on constructing the optimal $H_\infty$ controller from the approach of weighted $H_2$ optimization in Eq. (3). The determination of the frequency-dependent weight $W$ in Eq. (3) depends on the structure of $F_l(P, K)$. Two special cases have been considered in the literature. Kwakernaak[11] and Grimble[12] obtained $W$ for the mixed sensitivity problem

$$F_l(P, K) = \begin{bmatrix} W_SS \\ W_RR \end{bmatrix} \tag{6a}$$

where $S(s) = [I + G(s)K(s)]^{-1}$ is the output sensitivity matrix and $R(s) = K(s)[I + G(s)K(s)]^{-1}$ is the control sensitivity matrix. The problem considered by Kwakernaak[6] is for the case

$$F_l(P, K) = \begin{bmatrix} W_SS \\ W_TT \end{bmatrix} \tag{6b}$$

where $T(s) = G(s)K(s)[I + G(s)K(s)]^{-1}$ is the complementary sensitivity function. Their approach first determined the frequency-dependent weight $W$ according to the all-pass condition (2), and then obtained the optimal $H_\infty$ controller by solving an auxiliary LQG problem with dynamic weighting matrices, i.e., by solving an auxiliary weighted $H_2$-optimization problem. With their approach, the formulation strongly depends on the structure of $F_l(P, K)$, and the general procedure of finding the desired $H_2$ frequency-dependent weight $W$ for an arbitrary $F_l(P, K)$ is still lacking in the literature.

In this section we propose a loop-shaping design procedure to construct $W$ and the corresponding optimal $H_\infty$ controller for arbitrary structure of $F_l(P, K)$. The ultimate loop shape we want to achieve is a uniform frequency response of the largest singular value of $F_l(P, K)$, i.e., the loop shape represented by the all-pass condition (2). The strategy we will adopt is to shape $W$ recursively by a sequence of $H_2$ optimizations until a uniform shape of $\bar{\sigma}[F_l(P, K)(j\omega)]$ is achieved. Notice that although we focus on the uniformity of $\bar{\sigma}[F_l(P, K)(j\omega)]$, the shape of $\bar{\sigma}[S(j\omega)]$ and $\bar{\sigma}[T(j\omega)]$ can be assigned via proper choice of the weighting function $W(s)$. Taking mixed sensitivity problem (6b) for instance, when $\bar{\sigma}[F_l(P, K)(j\omega)]$ becomes uniform, the magnitudes of $\bar{\sigma}[S(j\omega)]$ and $\bar{\sigma}[T(j\omega)]$ are approximately proportional to the inverse magnitudes of $\bar{\sigma}[W_S(j\omega)]$ and $\bar{\sigma}[W_T(j\omega)]$, respectively. Hence, through proper choices of $W_S$ and $W_T$ we can impose the desired constraints on the singular values of $S$ and $T$.

The underlying concept of the $H_2$-based loop shaping design is easy to comprehend. For a scalar frequency-dependent weight $W(s)$, we have the following identity:

$$\bar{\sigma}[WF_l(P, K)(j\omega)] = |W(j\omega)|\bar{\sigma}[F_l(P, K)(j\omega)] \tag{7}$$

It can be rewritten in an alternative way as

$$|W(j\omega)| = \frac{\bar{\sigma}[WF_l(P, K)(j\omega)]}{\bar{\sigma}[F_l(P, K)(j\omega)]} \tag{8}$$

where $K$ is an optimal $H_2$ controller. Since both $W(s)$ and $K(s)$ are unknown, we can neither determine $W(s)$ from Eq. (8), nor can we determine $K(s)$ from the optimization of $\|WF_l(P, K)\|_2$; however, if we apply Eqs. (7) and (8) to different iteration steps, $K(s)$ and $W(s)$ can be determined iteratively from each other. For example, if we have a sequence of optimal $H_2$ controllers $(K_0, K_1, \ldots, K_i, \ldots)$ where we suppose $K_0(s)$ to $K_{i-1}(s)$ and the corresponding frequency-dependent weights $W_0(s)$ to $W_{i-1}(s)$ are known. The objective is to determine $K_i(s)$ and $W_i(s)$. We can exploit $K_{i-1}(s)$ and $W_{i-1}(s)$ to determine $W_i(s)$ by applying Eq. (8):

$$|W_i(j\omega)| = \frac{\bar{\sigma}[W_{i-1}F_l(P, K_{i-1})(j\omega)]}{\|F_l(P, K_{i-1})\|_\infty} \tag{9}$$

where we have replaced $\bar{\sigma}[F_l(P, K_{i-1})(j\omega)]$ in Eq. (8) by $\|F_l(P, K_{i-1})\|_\infty$, since if $K_{i-1}(s)$ is close to the optimal $H_\infty$ controller, we must have $\bar{\sigma}[F_l(P, K_{i-1})(j\omega)] = \|F_l(P, K_{i-1})\|_\infty, \forall\omega$. Next, we use this $W_i(s)$ to determine the optimal $H_2$ controller

$$K_i = \arg \inf_{K \text{ stabilizing}} \|W_iF_l(P, K)\|_2 \tag{10}$$

As the iteration proceeds, the following sequences can be defined:

$$\lambda_i = \|W_i F_l(P, K_i)\|_2 \tag{11}$$

$$\zeta_i(\omega) = \bar{\sigma}[F_l(P, K_i)(j\omega)] \tag{12}$$

$$\gamma_i(\omega) = \bar{\sigma}[W_i F_l(P, K_i)(j\omega)] \tag{13}$$

where $W_i(s)$ is a scalar, minimum-phase transfer function whose magnitude is determined by $\gamma_{i-1}(\omega)/\|\zeta_{i-1}(\omega)\|_\infty$, i.e.,

$$|W_i(j\omega)| = \frac{\gamma_{i-1}(\omega)}{\|\zeta_{i-1}(\omega)\|_\infty}, \qquad i = 1, 2, \ldots \tag{14}$$

with $|W_0(j\omega)| = 1$. The sequence of the optimal $H_2$ controllers converging to an optimal $H_\infty$ controller can be characterized in the following way.

*Theorem 1:* Given the controller sequence $(K_i)_{i=0}^\infty$ defined by Eq. (10) with $W_i$ given by Eq. (14), then the corresponding sequences $(\lambda_i)_{i=0}^\infty$, $[\zeta_i(\omega)]_{i=0}^\infty$, $[\gamma_i(\omega)]_{i=0}^\infty$ possess the following properties.

Property 1:

$$(1/\sqrt{n})\lambda_i \le \|\gamma_i(\omega)\|_2 \le \lambda_i \tag{15}$$

Property 2:

$$\frac{\zeta_{i+1}(\omega)}{\|\zeta_i(\omega)\|_\infty}\gamma_i(\omega) = \gamma_{i+1}(\omega), \qquad \forall\omega \tag{16}$$

Property 3:

$$|W_{i+1}(j\omega)| = \frac{\gamma_i(\omega)}{\|\zeta_i(\omega)\|_\infty} = \prod_{k=0}^{i} \frac{\zeta_k(\omega)}{\|\zeta_k(\omega)\|_\infty}, \qquad \forall\omega \tag{17}$$

where $n$ is the rank of $W_i(j\omega)F_l(P, K_i)(j\omega)$.

*Proof:*

1) Exploiting the relation

$$\frac{1}{n}\sum_{k=1}^{n}\sigma_k^2[W_i F_l(P, K_i)(j\omega)] \le \sigma_1^2[W_i F_l(P, K_i)(j\omega)] \le \sum_{k=1}^{n}\sigma_k^2[W_i F_l(P, K_i)(j\omega)]$$

and integrating with respect to $\omega$ from $-\infty$ to $\infty$, we have

$$\sqrt{\frac{1}{n}\int_{-\infty}^{\infty}\sum_{k=1}^{n}\sigma_k^2[W_i(j\omega)F_l(P, K_i)(j\omega)]\,d\omega} \le \sqrt{\int_{-\infty}^{\infty}\sigma_1^2[W_i(j\omega)F_l(P, K_i)(j\omega)]\,d\omega} \le \sqrt{\int_{-\infty}^{\infty}\sum_{k=1}^{n}\sigma_k^2[W_i(j\omega)F_l(P, K_i)(j\omega)]\,d\omega}$$

From the definition of the $H_2$ norm, these inequalities lead to the desired result,

$$(1/\sqrt{n})\|W_i F_l(P, K_i)\|_2 \le \|\sigma_1[W_i F_l(P, K_i)]\|_2 \le \|W_i F_l(P, K_i)\|_2$$

2) From the definition of $\gamma_{i+1}(\omega)$, we have

$$\gamma_{i+1}(\omega) = \bar{\sigma}[W_{i+1}(j\omega)F_l(P, K_{i+1})(j\omega)] = |W_{i+1}(j\omega)|\bar{\sigma}[F_l(P, K_{i+1})(j\omega)] = \frac{\gamma_i(\omega)}{\|\zeta_i(\omega)\|_\infty}\zeta_{i+1}(\omega)$$

3) By applying Eq. (16) repeatedly, we have

$$|W_{i+1}(\omega)| = \frac{\gamma_i(\omega)}{\|\zeta_i(\omega)\|_\infty} = \frac{\zeta_i(\omega)}{\|\zeta_i(\omega)\|_\infty}\frac{\gamma_{i-1}(\omega)}{\|\zeta_{i-1}(\omega)\|_\infty} = \frac{\zeta_i(\omega)}{\|\zeta_i(\omega)\|_\infty}\frac{\zeta_{i-1}(\omega)}{\|\zeta_{i-1}(\omega)\|_\infty}\frac{\gamma_{i-2}(\omega)}{\|\zeta_{i-2}(\omega)\|_\infty} \\ \vdots \\ = \prod_{k=0}^{i}\frac{\zeta_k(\omega)}{\|\zeta_k(\omega)\|_\infty}$$

QED

Now we are ready to show that the limit controller $K_\infty$ in the sequence of the optimal $H_2$ controllers $(K_i)_{i=0}^\infty$ defined in Eq. (10) is the solution of the $H_\infty$-optimization problem stated in Eq. (1). The proof contains three steps.

1) The first step is to show the convergence of the sequences $(\lambda_i)_{i=0}^\infty$ and $[\gamma_i(\omega)]_{i=0}^\infty$.

2) The second step is to show that the limit controller $K_\infty$ satisfies the all-pass property: $\bar{\sigma}[F_l(P, K_\infty)(j\omega)] = \zeta_\infty, \forall\omega$.

3) The third step is to show that the limit controller $K_\infty$ truly achieves the infimum:

$$\inf_K \|F_l(P, K)\|_\infty = \|F_l(P, K_\infty)\|_\infty = \zeta_\infty$$

The convergence of the sequences $(\lambda_i)_{i=0}^\infty$ and $[\gamma_i(\omega)]_{i=0}^\infty$ is proved first.

*Theorem 2:*

1) The sequence $(\lambda_i)_{i=0}^\infty$ is convergent.

2) Let $K_i$ be the central solution[14] of the $H_2$-optimization problem defined in Eq. (10); then the controller sequence $(K_i)_{i=0}^\infty$ is convergent.

3) The function sequence $[\gamma_i(\omega)]_{i=0}^\infty$ is convergent.

*Proof:*

1) By definition [from Eq. (17)],

$$\lambda_i = \left\|\prod_{k=0}^{i-1}\frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty}F_l(P, K_i)\right\|_2 = \left\|\frac{\zeta_i(\omega)}{\|\zeta_i\|_\infty}\right\|_\infty\left\|\prod_{k=0}^{i-1}\frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty}F_l(P, K_i)\right\|_2 \qquad \left(\text{note } \left\|\frac{\zeta_i(\omega)}{\|\zeta_i\|_\infty}\right\|_\infty = 1\right)$$

$$\ge \left\|\prod_{k=0}^{i}\frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty}F_l(P, K_i)\right\|_2 \ge \inf_K\left\|\prod_{k=0}^{i}\frac{\zeta_k(\omega)}{\|\zeta_k\|_\infty}F_l(P, K)\right\|_2 = \lambda_{i+1}$$

Hence, the sequence $(\lambda_i)_{i=0}^\infty$ is monotonically decreasing and bounded by $0 \le \lambda_i \le \lambda_0$. Employing the convergent theorem[15] of real sequences, we thus obtain the convergence of $(\lambda_i)_{i=0}^\infty$.

2) The solution $K_i$ of the optimal $H_2$-optimization problem (10) is, in general, not unique. Nevertheless, if we confine $K_i$ to be the central solution derived in Ref. 14, the one-to-one correspondence between $K_i$ and $\lambda_i$ can be established. Under such circumstance, the convergence of $\lambda_i$ implies the convergence of $K_i$.

3) From the definition, we know that the sequence $[\gamma_i(\omega)]_{i=0}^\infty$ is uniquely determined by the controller sequence $(K_i)_{i=0}^\infty$. Therefore, the convergence of $\gamma_i(\omega)$ is guaranteed by the convergence of $K_i$.

QED

Having proved the convergence of the two related sequences, we proceed to verify the uniformity of the largest singular value of $F_l(P, K_\infty)(j\omega)$.

*Lemma 1:* Let $K_\infty$ be the limit of the controller sequence defined in Eq. (10); then

$$\bar{\sigma}[F_l(P, K_\infty)(j\omega)] = \|\zeta_\infty(\omega)\|_\infty = \text{const} \qquad \forall\omega \tag{18}$$

*Proof:* From Eq. (16), we have

$$\lim_{i\to\infty} \frac{\zeta_i(\omega)}{\|\zeta_{i-1}(\omega)\|_\infty} = \lim_{i\to\infty} \frac{\gamma_i(\omega)}{\gamma_{i-1}(\omega)}$$

Recall the convergence of $\gamma_i(\omega)$, we obtain $\lim_{i\to\infty}\gamma_i(\omega)/\gamma_{i-1}(\omega) = 1$, $\forall\omega$. It turns out that $\lim_{i\to\infty} \zeta_i(\omega)/\|\zeta_{i-1}(\omega)\|_\infty = 1 \forall\omega$, that is, $\zeta_\infty(\omega) = \bar{\sigma}[F_l(P, K_\infty)(j\omega)] = \|\zeta_\infty(\omega)\|_\infty = \text{const}, \forall\omega$. QED

The final step in our characterization of the optimal $H_\infty$ controller via $H_2$-based loop-shaping design is to show that $K_\infty$ achieves the infimum: $\inf_K \|F_l(P, K)\|_\infty = \|F_l(P, K_\infty)\|_\infty$. The justification of using the optimal $H_2$ controllers to synthesize the optimal $H_\infty$ controllers mainly relies on the following theorem.

*Theorem 3:* Suppose there exists a scalar frequency-dependent weight $W(s)$ such that the optimal $H_2$ controller $K_0$ obtained from

$$K_0 = \arg\inf_K \|W F_l(P, K)\|_2 \tag{19}$$

satisfies the all-pass condition: $\bar{\sigma}[F_l(P, K_0)(j\omega)] = \zeta = \text{const}, \forall\omega$, then $K_0$ minimizes the $H_\infty$-norm criterion: $\inf_K \|F_l(P, K)\|_\infty = \|F_l(P, K_0)\|_\infty = \zeta$.

*Proof:* We will prove by contradiction. Assume there exists a controller $K_0'$ satisfying

$$\|F_l(P, K_0')\|_\infty < \zeta = \bar{\sigma}[F_l(P, K_0)(j\omega)], \qquad \forall\omega \tag{20}$$

then we have

$$\begin{aligned} \|W F_l(P, K_0')\|_2 &\le \|W\|_2 \|F_l(P, K_0')\|_\infty \\ &< \zeta\|W\|_2 = \|\zeta W\|_2 \\ &= \|W(j\omega)\bar{\sigma}[F_l(P, K_0)(j\omega)]\|_2 \\ &= \|\bar{\sigma}[W F_l(P, K_0)(j\omega)]\|_2 \end{aligned} \tag{21}$$

By the definition of the $H_2$ norm,

$$\|\bar{\sigma}[W F_l(P, K_0)(j\omega)]\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma_1^2[W F_l(P, K_0)(j\omega)]\,\mathrm{d}\omega}$$

$$\|W F_l(P, K_0)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{k=1}^{n} \sigma_k^2[W F_l(P, K_0)(j\omega)]\,\mathrm{d}\omega}$$

It follows that $\|\bar{\sigma}[W F_l(P, K_0)(j\omega)]\|_2 \le \|W F_l(P, K_0)\|_2$. Using this result in Eq. (21), we have $\|W F_l(P, K_0')\|_2 < \|W F_l(P, K_0)\|_2$. But this violates the fact in Eq. (19). Hence, the assumption made in Eq. (20) is incorrect, i.e., the infimum is truly achieved by $K_0$. QED

In Lemma 1 we have shown that the limit controller $K_\infty$ obtained from $K_\infty = \arg\inf_K \|W_\infty F_l(P, K)\|_2$ satisfies the all-pass condition (18). Hence from Theorem 3, we recognize $K_\infty$ as the desired optimal $H_\infty$ controller, and the associated frequency-dependent weight $W(s)$ in Theorem 3 is $W_\infty(s)$, which is the limit of the sequence $(W_i)_{i=0}^\infty$ defined in Eq. (14).

## $H_2$-Based $H_\infty$-Synthesis Algorithm

Using Eq. (17), the controller sequence $(K_i)_{i=0}^\infty$ defined in Eq. (10) can be rewritten in the following manner:

$$\begin{aligned} K_i &= \arg\inf_K \|W_i F_l(P, K)(j\omega)\|_2 \\ &= \arg\inf_K \left\| \prod_{k=0}^{i-1} \frac{\zeta_k(\omega)}{\|\zeta_k(\omega)\|_\infty} F_l(P, K)(j\omega) \right\|_2 \\ &= \arg\inf_K \left\| \prod_{k=0}^{i-1} \zeta_k(\omega) F_l(P, K)(j\omega) \right\|_2 \end{aligned} \tag{22}$$

If we define a new scalar minimum-phase function $\hat{W}_i$ as

$$|\hat{W}_i(j\omega)| = \prod_{k=0}^{i-1} \zeta_k(\omega) \tag{23}$$

then Eq. (22) becomes $K_i = \arg\inf_K \|\hat{W}_i F_l(P, K)\|_2$. Instead of the frequency-dependent weight $W_i(s)$ defined in Eq. (14), we can use $\hat{W}_i(s)$ as a new frequency-dependent weight. The following work is to derive the recursive formula for $\hat{W}_i$:

$$\begin{aligned} |\hat{W}_{i+1}(j\omega)| &= \prod_{k=0}^{i} \zeta_k(\omega) = \left(\prod_{k=0}^{i-1} \zeta_k(\omega)\right)\zeta_i(\omega) \\ &= |\hat{W}_i(j\omega)|\bar{\sigma}[F_l(P, K_i)(j\omega)] \\ &= \bar{\sigma}[\hat{W}_i F_l(P, K_i)(j\omega)] \end{aligned} \tag{24}$$

In terms of the new frequency-dependent weight $\hat{W}_i(s)$, we summarize the $H_\infty$-synthesis technique using $H_2$-based loop-shaping procedures in the following algorithm.

Initialization: $i = 0$.

1) Set $\hat{W}_0(s) = 1$.

2) Compute $K_0 = \arg\inf_K \|\hat{W}_0 F_l(P, K)\|_2$ using $H_2$-optimization technique.

3) Compute $\bar{\sigma}[\hat{W}_0 F_l(P, K_0)(j\omega)]$.

Recursive formula: $i = 1, 2, 3, \ldots$.

1) Set $\deg(\hat{W}_i) = n_w$, where $\hat{W}_i$ is a scalar, minimum-phase function.

2) Fit $\bar{\sigma}[\hat{W}_{i-1} F_l(P, K_{i-1})(j\omega)]$ by $|\hat{W}_i(j\omega)|$ [from Eq. (24)].

3) Compute $K_i = \arg\inf_K \|\hat{W}_i F_l(P, K)\|_2$.

4) Compute $\bar{\sigma}[\hat{W}_i F_l(P, K_i)(j\omega)]$.

Repeat the algorithm until the required accuracy in the uniformity of $\zeta_i(\omega) = \bar{\sigma}[F_l(P, K_i)(j\omega)]$ is met. Ideally, at the end of iteration $\bar{\sigma}[F_l(P, K_\infty)(j\omega)]$ must be an all-pass function, as shown in Lemma 1. This algorithm is a new scheme for optimal $H_\infty$ synthesis. Although the convergence of the sequence to the optimal $H_\infty$ controller is guaranteed theoretically, in numerical calculation, the achievable degree of optimization depends on the accuracy of calculating the optimal $H_2$ controller and on the accuracy of curve fitting. The order of the $H_2$ controllers in the minimizing sequence can be chosen a priori according to the required closeness to the optimal $H_\infty$ solution. Once controller order is assigned, the order $n_w$ of $\hat{W}_i$ in curve fitting can be determined accordingly.

The multiplication of $F_l(P, K_i)$ by a scalar function $\hat{W}_i$ can be considered as a shaping effect on the augmented plant $P$ by noting that $\hat{W}_i F_l(P, K) = F_l(P_i, K)$, where

$$P_i(s) = \begin{bmatrix} \hat{W}_i(s) I_l & 0 \\ 0 & I_m \end{bmatrix} P$$

with $l$ and $m$ being the dimensions of the output vector $z$ and the measurement vector $y$, respectively. In this way, a scalar weighting function, instead of matrix-valued weighting function, can always be used effectively, regardless of the dimension of the augmented plant.

In summary, the procedures of synthesizing the optimal $H_\infty$ controllers via the $H_2$-based loop-shaping design involve only two steps: one step is the refinement of the frequency-dependent weight via $|\hat{W}_i(j\omega)| = \bar{\sigma}[\hat{W}_{i-1} F_l(P, K_{i-1})(j\omega)]$; the other step is the refinement of the optimal $H_2$ controller via $K_i(s) = \arg\inf_K \|\hat{W}_i F_l(P, K)\|_2$.

## Numerical Examples

These $H_2$-based loop-shaping procedures are applied to design $H_\infty$ controllers for two real plants, a modern fighter[9,16] and a large space structure.[16,17] These two standard benchmark problems, which have been solved in detail by conventional $\gamma$-iteration approach, provide the comparison basis between the present method and the $\gamma$-iteration technique.

**Fighter $H_\infty$ Design Example**

This section contains an example of $H_\infty$ synthesis as applied to the pitch-axis controller design of an experimental highly maneuverable airplane HIMAT.[9] The linearized model of HIMAT consists of six states: $x^T = (\delta v, \alpha, q, \theta, \alpha_1, \alpha_2)$, representing the forward velocity, angle of attack, pitch rate, pitch angle, and actuator states, respectively. The control inputs are elevon $\delta_e$ and canard $\delta_c$. The variables to be measured are $\alpha$ and $\theta$. The state-space nominal model for this two-input two-output plant is given by

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$$= \left[\begin{array}{cccccc|cc} -0.023 & -37 & -19 & -32 & 3.3 & -0.76 & 0 & 0 \\ 0 & -1.9 & 0.98 & 0 & -0.17 & 0 & 0 & 0 \\ 0.012 & 12 & -2.6 & 0 & -32 & 22 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30 & 0 & 30 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

The singular value design specifications are as follows.

1) The robustness specification is −40 dB/decade roll-off and at least −20 dB at 100 rad/s. The weighting function $W_S$ reflecting this specification can be chosen as[16]

$$W_S(s) = 16.8 \begin{bmatrix} \dfrac{s+100}{100s+1} & 0 \\ 0 & \dfrac{s+100}{100s+1} \end{bmatrix}$$

2) For the performance specification, minimize the sensitivity function as much as possible. The associated weighting function $W_T$ is chosen as[16]

$$W_T(s) = \begin{bmatrix} \dfrac{s^2}{1000} & 0 \\ 0 & \dfrac{s^2(\tau s+1)}{1000} \end{bmatrix}$$

where $\tau = 0.5$ ms is selected such that both channels are penalized equally up to $1/\tau$ rad/s. Note that because $W_T$ is an improper transfer function, it cannot be realized in state-space form; but $W_T(s)G(s)$ is proper. This particular $W_T(s)$ ensures that the $D_{12}$ matrix of the augmented plant $P(s)$ is full rank. Augment the nominal plant $G(s)$ with the weighting functions $W_S(s)$ and $W_T(s)$ to form the augmented plant $P(s)$ as

$$P(s) = \left[\begin{array}{c|c} W_S & -W_S G \\ 0 & W_T G \\ \hline I & -G \end{array}\right]$$

and the $H_\infty$-optimization problem is to find $K$ to minimize $\|F_l(P, K)\|_\infty$. The interconnection of the frequency-dependent weight $\hat{W}_i(s)$ with the original system is demonstrated in Fig. 3. We will shape the weighting functions $W_S(s)$ and $W_T(s)$ such that the $H_\infty$-optimization problem can be converted to an equivalent weighted $H_2$-optimization problem with the reshaped weighting functions given by $\hat{W}_i(s)W_S(s)$ and $\hat{W}_i(s)W_T(s)$. Different order of $\hat{W}_i$ and different number of iterations can be exploited to test the accuracy and the converging tendency of the proposed scheme. For example, we use fourth-order $\hat{W}_i(s)$ for curve fitting. Four iterations have been performed and yield five optimal $H_2$ controllers:

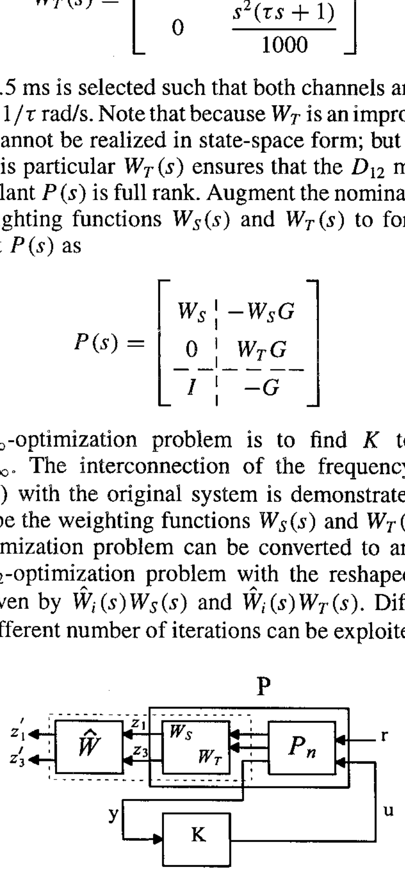


**Fig. 3 Block diagram with auxiliary weighting function $\hat{W}(s)$.**

Table 1 Fighter example results

| No. | $\|\hat{W}_i F_l(P, K_i)\|_2$ | $\|F_l(P, K_i)\|_\infty$ |
|---|---|---|
| 0 | 7.1032 | 1.4830 |
| 1 | 0.9657 | 1.0827 |
| 2 | 0.9643 | 0.9813 |
| 3 | 0.9650 | 0.9727 |
| 4 | 0.9648 | 0.9663 |

$$K_i = \arg\inf_K \|\hat{W}_i F_l(P, K)\|_2, \qquad i = 0, 1, 2, 3, 4 \tag{25}$$

Numerical results for $\|\hat{W}_i F_l(P, K_i)\|_2$ and $\|F_l(P, K_i)\|_\infty, i = 0, 1, 2, 3, 4$ are shown in Table 1.

The plot of $\zeta_i(\omega) = \bar{\sigma}[F_l(P, K_i)(j\omega)]$ is shown in Fig. 4, where we can observe that $K_4$ is very close to the optimal $H_\infty$ controller rendering $\bar{\sigma}[F_l(P, K_4)(j\omega)] = \text{const}, \forall\omega$. The weighting functions $W_S$ and $W_T$ before and after reshaping by $\hat{W}_4$ are shown in Figs. 5 and 6, respectively. If we fit $\hat{W}_i$ with higher degrees and increase the computational accuracy, the sequence of the optimal $H_2$ controllers $K_i$ can be made arbitrarily close to the optimal $H_\infty$ solution. This example illustrates, as proved earlier, that by recursive frequency shaping, the optimal $H_2$ controllers can converge gradually to the optimal $H_\infty$ controller. From the viewpoint of numerical computation, the main advantage of the present $H_\infty$ synthesis technique is its fast convergent speed with reduced computational effort, when compared with the $\gamma$-iteration technique. The differences between these two schemes are listed next.

1) To start $\gamma$ iteration, we need an initial guess for the $H_\infty$ norm. Hence, the required iteration number depends on the closeness between the initial guess and the optimal solution. However, no initial guess is required in the present scheme.

2) The accuracy of the $\gamma$-iteration scheme depends on the error tolerance set by the users. The error tolerance stands for the difference of the $\gamma$ values between the two consecutive iterations. For instance, if we set the error tolerance to 0.01, the typical required iteration number is about 10, whereas if we reduce the error tolerance to 0.001, the typical required iteration number is about 20. However, the accuracy of the present scheme does not have as strong a dependence on the iteration number as that of the $\gamma$ iteration scheme. In the many case studies conducted by the authors, the present scheme appears to achieve its steady state after four or five iterations. Iteration after five is often unnecessary, since the output of the algorithm does not change within the given error tolerance. The closeness of the steady-state value to the optimal $H_\infty$ norm depends on the accuracy of curve fitting. In short, the accuracy of the $\gamma$-iteration scheme is dominated by the number of iterations, whereas the accuracy of the present scheme is dominated by the accuracy of curve fitting technique.

3) At each iteration, the $\gamma$-iteration scheme solves a suboptimal $H_\infty$ control problem, whereas the present scheme solves an optimal $H_2$ control problem. These two control problems[14] are governed by two similar pairs of algebraic Riccati equations (ARE). The computational time to solve the $H_\infty$ AREs and the $H_2$ AREs is quite the same.

For the purpose of the comparison, the example is resolved by the $\gamma$-iteration scheme. An initial guess of the optimal $H_\infty$ norm is required to start the $\gamma$-iteration process. The value $\lambda_0 = \|F_l(P, K_0)\|_\infty = 1.483$, with $K_0$ given by Eq. (25), serves as a good starting point for $\gamma$ iteration. The error tolerance is set to 0.01, and the $\gamma$-iteration process is completed after 11 iterations. The result is shown in Fig. 4, where we can see that the singular value plots of the two schemes are nearly indistinguishable. The $H_\infty$ norms achieved by the $\gamma$-iteration scheme and the present scheme are 0.9642 and 0.9663, respectively. However, although the resulting accuracy is comparable, the $\gamma$-iteration scheme achieves this accuracy by solving the $H_2$ AREs a single time (for initial guess) and the $H_\infty$ AREs 11 times, whereas the present scheme solves the $H_2$ AREs only 5 times.

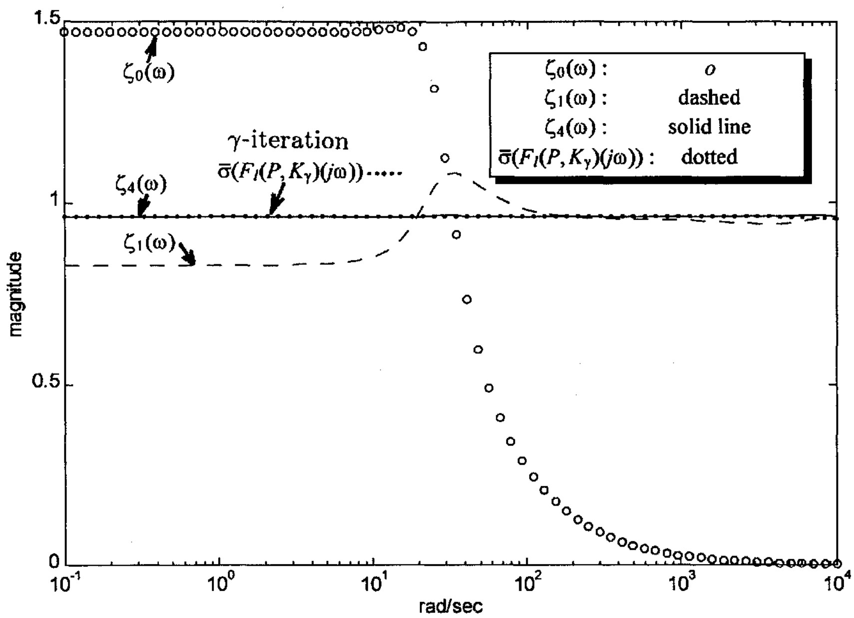


**Fig. 4 Iteration results of the fighter design example.**

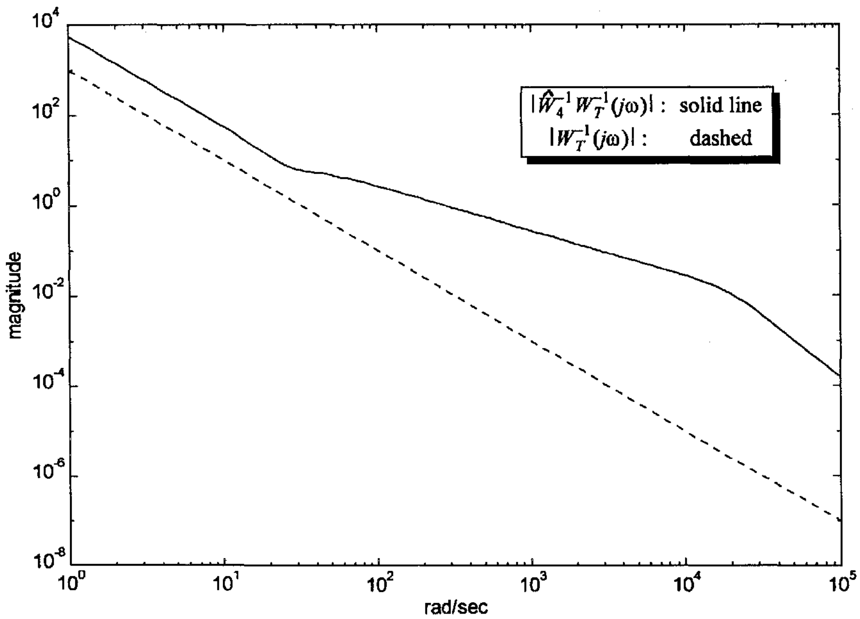


**Fig. 5 Reshaping of $W_T^{-1}(s)$ by $\hat{W}_4^{-1}(s)$ of fighter example.**

**Large Space Structure Design Example**

The large space structure (LSS) model[17] was generated by the NASTRAN finite element program by TRW Space Technology Group. It consists of 58 vibrational modes and is controlled by 18 actuators and 20 sensors. To simulate the real environmental vibration source, 12 disturbances are acting on the top and the bottom of the structure. This leads to a state-space representation of the form $\dot{x} = Ax + Bu$, $y = Cx$, where $A \in R^{116\times 116}$, $B \in R^{116\times 30}$, and $C \in R^{20\times 116}$. The four-state approximation of the plant with square-down filter is given by

$$G(s) = \left[\begin{array}{cccc|cc} -0.9900 & 0.0005 & 0.4899 & 1.9219 & 0.7827 & -0.6140 \\ 0.0009 & -0.9876 & 1.9010 & -0.4918 & 0.6130 & 0.7826 \\ -0.4961 & -1.9005 & -311.70 & 4.9716 & 0.7835 & 0.5960 \\ -1.9215 & 0.4907 & -7.7879 & -398.31 & 0.6069 & -0.7878 \\ \hline 0.7829 & 0.6128 & -0.7816 & -0.6061 & 0 & 0 \\ -0.6144 & 0.7820 & -0.5984 & 0.7884 & 0 & 0 \end{array}\right]$$

The LSS design specification requires the line-of-sight error to be attenuated at least 100:1 at frequency from 0 to 15 Hz after the feedback control loop is closed. Allowing for a 30 dB per decade roll-off beyond 15 Hz places the control loop bandwidth of roughly 300 Hz. These specifications lead to the following weighting functions.[16]

1) The robustness specification is −20 dB/decade roll-off above 2000 rad/s:

$$W_T(s) = \begin{bmatrix} \dfrac{s}{2000} & 0 \\ 0 & \dfrac{s}{2000} \end{bmatrix}$$

2) For the performance specification, minimize the sensitivity function

$$W_S(s) = 1.4 \begin{bmatrix} \dfrac{(1+s/5000)^2}{0.01(1+s/100)^2} & 0 \\ 0 & \dfrac{(1+s/5000)^2}{0.01(1+s/100)^2} \end{bmatrix}$$

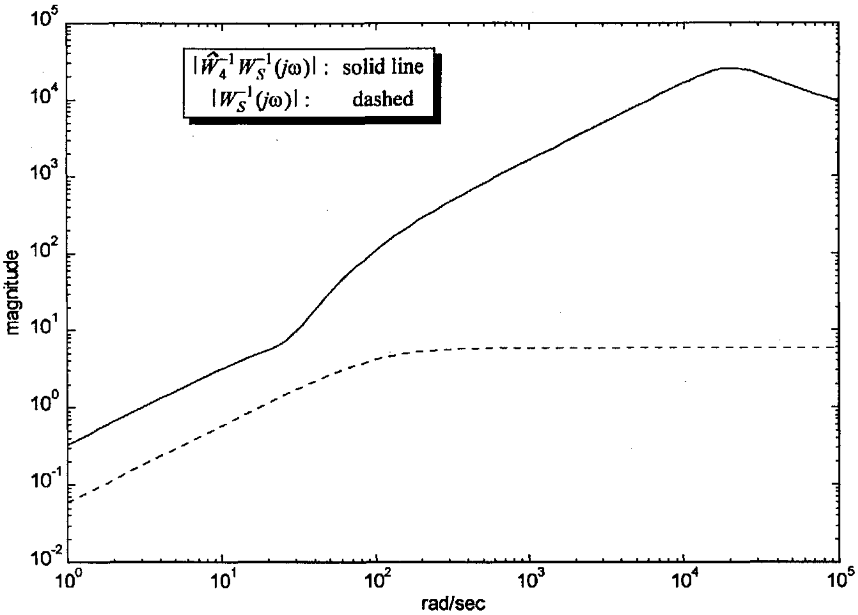


Fig. 6 Reshaping of $W_S^{-1}(s)$ by $\hat{W}_4^{-1}(s)$ of fighter example.

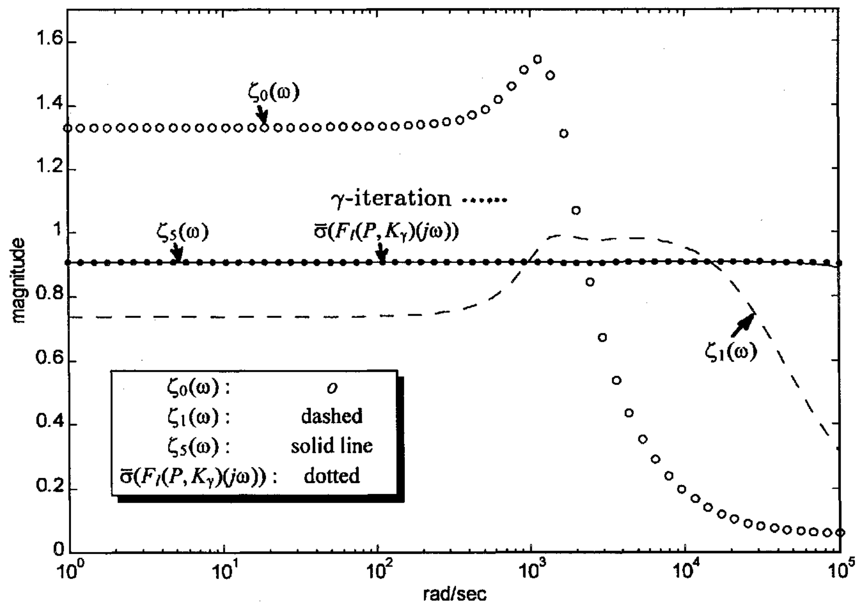


Fig. 7 Iteration results of large space structure design example.

Table 2 LSS example results

| No. | $\|\hat{W}_i F_l(P, K_i)\|_2$ | $\|F_l(P, K_i)\|_\infty$ |
|---|---|---|
| 0 | 114.41 | 1.5445 |
| 1 | 0.4916 | 0.9915 |
| 2 | 0.9106 | 0.9317 |
| 3 | 0.9139 | 0.9176 |
| 4 | 0.8999 | 0.9089 |
| 5 | 0.9068 | 0.9074 |

After augmentation with $W_S$ and $W_T$, the augmented plant $P(s)$ has eight states. This design problem is reduced to the minimization of $\|F_l(P, K)\|_\infty$, which, in turn, is solved by the $H_2$-based loop-shaping procedures proposed previously. After five iterations, we obtained the results given in Table 2.

The frequency responses of $\bar{\sigma}[F_l(P, K_i)(j\omega)]$ are depicted in Fig. 7, where we can see that $\bar{\sigma}[F_l(P, K_5)(j\omega)]$ keeps constant up to the frequency $10^5$ rad/s with the constant value ($H_\infty$ norm) given by 0.9074. Also shown in Fig. 7 is the result from the $\gamma$-iteration scheme, where the error tolerance is set to 0.01 and the initial guess of the optimal $H_\infty$ norm is given by $\lambda_0 = \|F_l(P, K_0)\|_\infty = 1.5445$. The $\gamma$-iteration process is completed after 12 iterations, and the resulting $H_\infty$ norm is 0.905. As in the previous example, this example reveals that the accuracy of the present scheme by solving the $H_2$ AREs 6 times is comparable to the accuracy of the $\gamma$-iteration scheme by solving the $H_2$ AREs 1 time and the $H_\infty$ AREs 12 times.

Because the weighted $H_2$-optimization problems can be solved by LQG/LTR design procedure,[10] we can also implement the proposed $H_2$-based loop-shaping design in terms of the LQG/LTR procedure to obtain the optimal $H_\infty$ controllers.

## Conclusions

The $H_\infty$-optimization problem has been solved using $H_2$-based loop-shaping formulation where the desired closed-loop shape to be pursued is a uniform frequency response of the largest singular value. Along this approach, we have verified the possibility that conventional LQG controllers, with appropriate selection of frequency-dependent weights, can become optimal $H_\infty$ controllers. By providing a systematic methodology of determining the frequency-dependent weights via loop-shaping procedures proposed here, we have derived, both theoretically and numerically, a sequence of $H_2$ controllers converging to the optimal $H_\infty$ controllers. Without going into the details of the profound $H_\infty$ control theory, the proposed workable algorithm provides an easy access to the optimal $H_\infty$ controllers for engineers who are familiar with the LQG or $H_2$ control design.

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